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The Graph on the Sphere

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Abstract: In this paper, we will introduce new type of graphs on the sphere and explain the properties of this graph. The incidence and adjacent matrices will be defined for this graph. Some differences between this new graph and the normal graph will be discussed.

Keywords: Adjacency, Graph, incidence, matrix, sphere

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1. Definitions and Background:

Terminolgies:

An abstract graph G is a diagram consisting of a finite non- empty set of the elements called vertices denoted by $V(G)$ together with a set of unordered pairs these elements called edges and denoted by $E(G)$. The set of vertices of the graph G is called the **vertex set** of G and the list of edges is called the **edge list** of G [2, 3]. Let v and w be two vertices of a graph, if v and w are joined by an edge e, then v and w are said to be **adjacent**. Also, v and w are said to be **incident** with e, and e is said to be **incident** with v and w. [1 , 4 , 5].

If G be a graph without loops, with n- vertices labeled **1,2,3,…,n.** The " **adjacency matrix** " $A(G)$ in the **n x n** matrix in which the entry in row i and column j is the number which denotes the adjacent of a vertex v^i with vertex v^j. let G be a graph without loops, with n-vertices labeled **1,2,3,…..,n** and m- edges labeled **1,2,3,…...,m**. The " **incidence matrix** " $I(G)$ is the **n x m** matrix in which the entry in row i and and column j is **1** if vertex i is incident with edge j and **0** otherwise.

A graph on sphere $\mathbf{S}^2$ is defined as $\bar{G} = (\bar{V}, \bar{E})$ where $\bar{V}$ is a set of vertices on a geodesic, $\bar{E}$ is the set of edges $e_i \in \bar{E}$, e_i determined by three vertices in one specific direction clockwise or anti-clockwise.

2. Main Results:

Now, we will get more close to some kinds of graphs on spheres and see some of their properties . Also the matrices which represent this new graph . Let us start with the following definition:

Simple Graph on Sphere:

A graph $G = (V(G) , E(G))$ consists of two finite sets: $V(G)$**,** the vertex set of the graph , often denoted by just V , which is a nonempty set of elements called vertices, and $E(G)$**,** the edge set of the graph, often denoted by just E, which is a possibly empty set of elements called edges. Such that each edge e in E is assigned by three vertices in clockwise direction. **SeeFig.(1).**

i.e $e_o = \{v_o , x , v_1\}$, $e_1 = \{x , v_1 , v_o\}$, $e_2 = \{v_1 , v_o , x\}$

Example 1:

Fig.(1) represent the new graph $\bar{G}$ where the vertex set $V(\bar{G})$ is the set (v_o, x , v_1) and the edge-list $E(\bar{G})$ consists of the pairs $(v_o v_1)x$, $(x v_o)\, v_1$, $(v_1 x) v_0$ where $(v_o v_1)x$ denote the edge between the two vertices v_o, v_1 along **x** . And, $(x\, v_o)v_1$ denote the edge between the two vertices x, v_o along v_1 and so on.

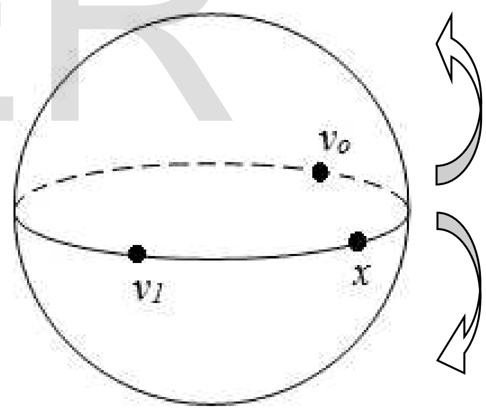

Fig.(1):A simple graph on sphere

$V(\bar{G}) = \{v_o , x , v_1\}$

and

$E(\bar{G}) = \{e_o = (v_o , x , v_1), e_1 = (x , v_1, v_o), e_2 = (v_1 , v_o , x)\}$

$= \{ e_o = (v_o v_1)_x , e_1 = (x\, v_o)_{v1} , e_2 = (v_1, x)_{vo}\}$

$= \{ e_o , e_1 , e_2 \}$

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Our work should be either in clockwise direction or anticlockwise direction one way is isomorphic, hence

Lemma:

There is an isomorphism between the clockwise movement in any simple graph on a sphere and anticlockwise movement.

The adjacency and incidence matrices of $\bar{G}$ are given by

$$A(\bar{G}) = \begin{array}{c} \\ v_0 \\ x \\ v_1 \end{array} \begin{array}{c} \begin{array}{ccc} v_0 & x & v_1 \end{array} \\ \begin{pmatrix} 0 & 1_{v_1} & 1_x \\ 1_{v_1} & 0 & 1_{v_0} \\ 1_x & 1_{v_0} & 0 \end{pmatrix} \end{array}$$

$$I(\bar{G}) = \begin{array}{c} \\ v_0 \\ x \\ v_1 \end{array} \begin{array}{c} \begin{array}{ccc} e_0 & e_1 & e_2 \end{array} \\ \begin{pmatrix} 1_x & 1_{v_1} & 0 \\ 0 & 1_{v_1} & 1_{v_0} \\ 1_x & 0 & 1_{v_0} \end{pmatrix} \end{array}$$

$1_{v_i}, 1_x$ ns the two vertices are the ends of the edge x respectively.

Example 2:

In the Fig. 2

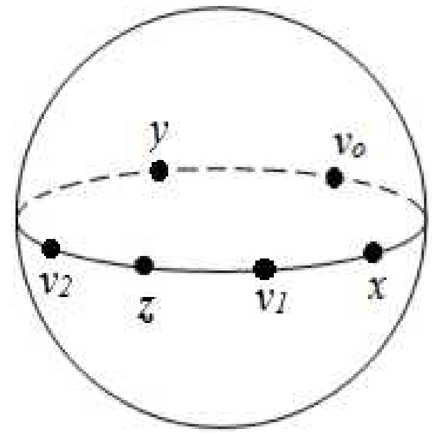

Fig2:The simple graph on sphere with six vertices

$V(\bar{G}) = \{v_o, x, v_1, z, v_2, y\}$

and

$E(\bar{G}) = \{e_o = \{v_o, x, v_1\}, e_1 = \{x, v_1, z\}, e_2 = \{v_1, z, v_2\},$
$e_3 = \{z, v_2, y\}, e_4 = \{v_2, y, v_o\}, e_5 = \{y, v_o, x\}\}$

$E(\bar{G}) = \{e_o = (v_o\, v_1)_x, e_1 = (x\, z)_{v1}, e_2 = (v_1 v_2)_z, e_3 = (z\, y)_{v2},$
$e_4 = (v_2 v_o)_y, e_5 = (y\, x)_{vo}\}$

$= \{e_o, e_1, e_2, e_3, e_4, e_5\}$

and the adjacency will be in the form:

$$A(\bar{G}) = \begin{array}{c} \\ v_0 \\ x \\ v_1 \\ z \\ v_2 \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} v_0 & x & v_1 & z & v_2 & y \end{array} \\ \begin{pmatrix} 0 & 0 & 1_x & 0 & 1_y & 0 \\ 0 & 0 & 0 & 1_{v_1} & 0 & 1_{v_0} \\ 1_x & 0 & 0 & 0 & 1_z & 0 \\ 0 & 1_{v_1} & 0 & 0 & 0 & 1_{v_2} \\ 1_y & 0 & 1_z & 0 & 0 & 0 \\ 0 & 1_{v_0} & 0 & 1_{v_2} & 0 & 0 \end{pmatrix} \end{array}$$

where $\mathbf{1_a}$ the two vertices are the ends of the edge along vertex **a.**

And the incident matrix is:

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$I(\bar{G}) =$

	e_0	e_1	e_2	e_3	e_4	e_5
v_0	1_x	0	0	0	1_y	0
x	0	1_{v_1}	0	0	0	1_{v_0}
v_1	1_x	0	1_z	0	0	0
z	0	1_{v_1}	0	1_{v_2}	0	0
v_2	0	0	1_z	0	1_y	0
y	0	0	0	1_{v_2}	0	1_{v_0}

Example 3:

Fig. 3 Consist of two great circles intersect by two vertices v_o and v_1 and ***u, n, k, m*** are the via vertices.

where:

$V(\bar{G}) = \{v_o, u, n, v_1, k, m\}$

and

$E(\bar{G}) = \{\{(v_o, u, v_1), (u, v_1, k), (v_1, k, v_o), (k, v_o, u), (v_o, n, v_1), (n, v_1, m), (v_1, m, v_o), (m, v_o, n)\}\}$

$= \{e_o, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

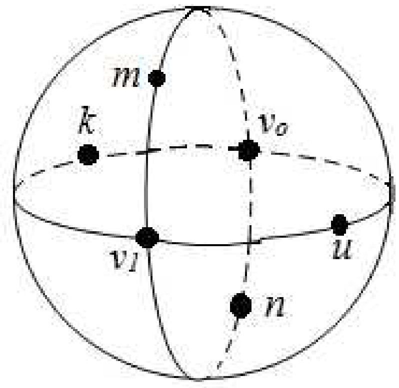

Fig3: Two simple graphs on sphere with six vertices

and the adjacency matrix will be in the form:

$A(\bar{G}) =$

	v_0	u	n	v_1	k	m
v_0	0	0	0	4	0	0
u	0	0	0	0	2	0
n	0	0	0	0	0	2
v_1	4	0	0	0	0	0
k	0	2	0	0	0	0
m	0	0	2	0	0	0

In the above. Matrix

4 in the first row represents $\mathbf{1_u + 1_k + 1_n + 1_m}$ as well as. **2** in the second row is equal to $\mathbf{1_{v1} + 1_{vo}}$ and **2** in the third row will equal $\mathbf{1_{v1} + 1_{vo}}$ Similarly **4, 2 , 2** in the fourth, fifth and sixth row of these matrix respectively .

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And the incident matrix is:

$I(\bar{G}) =$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_0	1_u	0	1_k	0	1_n	0	1_m	0
u	0	1_{v_1}	0	1_{v_0}	0	0	0	0
n	0	0	0	0	0	1_{v_1}	0	1_{v_0}
v_1	1_u	0	1_k	0	1_n	0	1_m	0
k	0	1_{v_1}	0	1_{v_0}	0	0	0	0
m	0	0	0	0	0	1_{v_1}	0	1_{v_0}

3. Types of the new graph:

3.1 Null Graph:

- It is a graph which contains only isolated nodes **"Vertices ", i.e.** the set of edges in a null graph is empty. **See Fig.(3.1.a).**

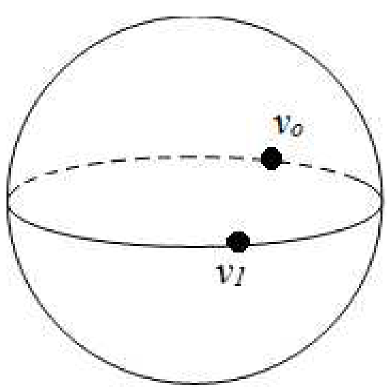

Fig 3.1.a: Null graph with two vertices only

- We may have null graphs either with two points or more . **i.e. E = ϕ See Fig.(3.1.b).**

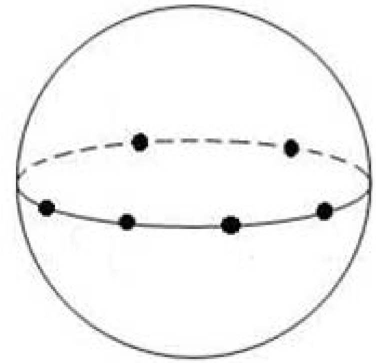

Fig 3.1.b: Null graph with alot of vertices

3.2 Loop:

An edge of a graph that joins a vertex to It self is called a loop. **See Fig.(3.2.a)**

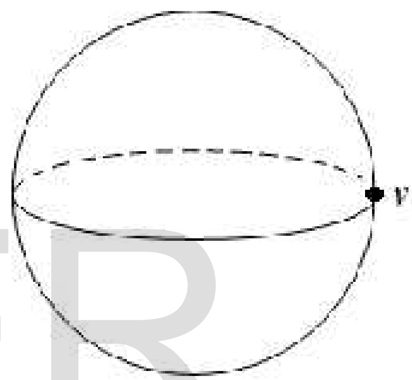

Fig 3.2.a: The loop on the sphere

- It's possible to draw infinite number of loops on the sphere .All loops are isomorphic in geometric sense.
- In similar normal graph it is possible to draw infinite number of loops at a single point .
- In two different points on the sphere we can draw two loops and these two loops intersect in exactly two points or no. **See Fig.(3.2.b)**

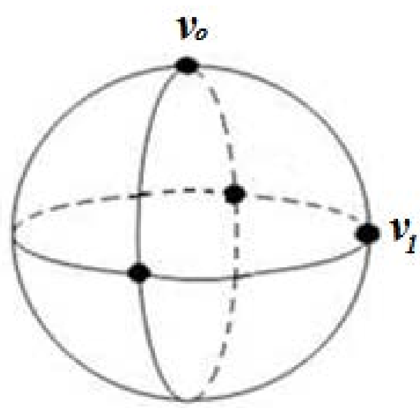

Fig 3.2.b: two loops from two vertices

4. Properties of the graph on a sphere:

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- An edge in the new graph is the curve passing through three vertices.
- There is only one great circle can pass through two certain points on the surface of the sphere.
- Any two great circles intersect in two points.
- Multigraph cannot be applied on the surface of the sphere, because only one great circle can pass through two certain points.

5. Notes:

- The largest circle in the sphere is called a great circle.
- The radius or (diameter) of the great circle is the radius or (diameter) of the sphere.
- Every great circle divides the sphere into two equal parts called hemispheres.
- A section in a sphere, S^2, is a S^1 curve and in case the radius of the section S^1 is the same of the radius of the sphere, S^2. Then this section is a great circle .
- Sections of the sphere that do not contain a diameter are called small circles. **See Fig (5.1)**

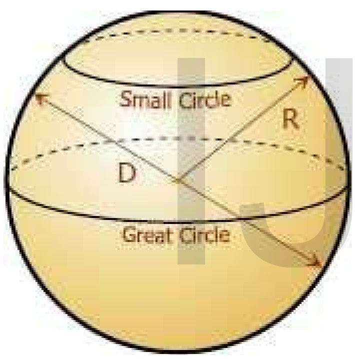

Fig 5.1: Small and great circles

Simple graphs can be drawn on small circles and in these cases isomorphism transforms these graphs into great circles.
i.e. All our work can be done in small circles as well.

6. Stereographic projection:

Now we defined the new graph it is the graph on the elliptic geometry by using the projection we can make a comparison between the graph on the sphere and the normal graph by using the Stereographic projection.

$V(X, Y, Z) \longrightarrow v(x, y, 0)$

$\infty \longrightarrow$ The north pole on the sphere.

Definition:

Let S^2 denote the unit sphere $X^2+Y^2+Z^2=1$ in the R^3

and let $N=(0, 0, 1)$ denote the "north pole" of S^2.

Given a point $P \in S^2$, other than N, then the line connecting N and P intersects the Xy - plane at a point $P^`$.

The **stereographic projection** is the map:

$$F : \left(S^2 - N\right) \rightarrow R^2 .$$

Fig. (6.1) is model of the Riemann sphere has its south pole resting on the origin of the complex plane . Each point on the surface of the Riemann sphere corresponds to a unique point in the complex plane and vice versa.

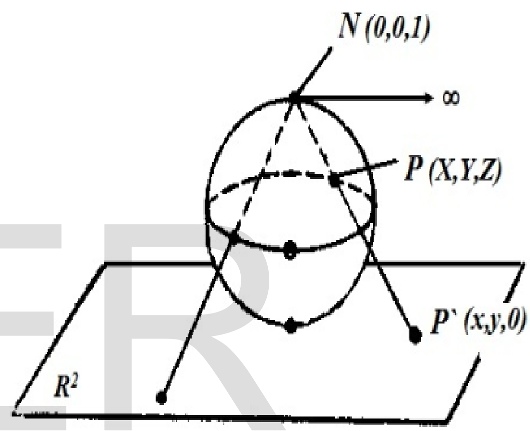

Fig 6.1: Riemann sphere

The fact that the points P, P' and N all lie on one line can be expressed by the fact that

$(X, Y, Z-1) = t(x, y, 1)$ (1)

for some non-zero real number t .
Here $P = (X, Y, Z)$, $N(0, 0, 1)$ and $P'(x, y, 0)$ the sphere of radius **1** with center at the origin is given by :

$G = \{(X, Y, Z) \,/\, X2 + Y2 + Z2 = 1\}$ (2)

An arbitrary plane in three-space is given by :

$AX + BY + CZ + D = 0$ (3)

for some arbitrary choice of the constants A, B, C and D. Thus a circle on the unit sphere is any set of points whose coordinates simultaneously satisfy equations *(2)* and *(3)*. The idea of the proof is that one can use equations *(2)* and *(1)* to write X as a function of t and x, Y as a function of t and , y and Z as a function of t and to simplify equation *(3)* to an equation in x and y . Since the equation in x and y

so obtained is clearly the equation of a circle in the xy-plan, the projection of the intersection of *(2)* and *(3)* is a circle.

To be more precise:

Equation *(1)* says that $X = tx$, $Y = ty$, and $1 - Z = t$.
set $Q = \frac{1+z}{1-z}$ and verify that

$z = \frac{Q-1}{Q+1}$, $1 + Q = \frac{2}{t}$

and $Q = x^2 + y^2$

If P lies on the plane,

$$AX + BY + CZ + D = 0$$

Thus

$$Atx + Bty + C\frac{Q-1}{Q+1} + D = 0$$

Or

$$\frac{2Ax}{Q+1} + \frac{2By}{Q+1} + C\frac{Q-1}{Q+1} + D = 0$$

Whence,

$$2Ax + 2By + C(Q-1) + D(Q+1) = 0$$

Since the coefficients of the x^2 and the y^2 terms are the same, this is the equation of a circle in the plane.

This shows that the new graph in which the edge is determined by three vertices turned

Or

$$(C + D)Q + 2Ax + 2By + D - C = 0$$

Recalling that

$$Q = x^2 + y^2$$

We see

$$(C + D)(x^2 + y^2) + 2Ax + 2By + D - C = 0 \qquad (4)$$

to the normal graph by **stereographic projection.**

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